We saw that the RF is a quasilinear parabolic PDE and is a nonlinear heat equation for a Riemannian metric.

Moreover, Rm, Ric, R etc. all satisfy heat type equations w/ reaction terms being quadratic. So, from the theory of parabolic PDEs of functions we expect that if we have bounds on the geometry of (M^n, g_0) then this would induce a priori bounds on the geometry of $g(t)$.

We see this here for $Rm(t)$. These are called Bernstein–Bando–Shi estimates.

<u>Theorem (Bando, Shi, Hamilton)</u>

Let $(M^n, g(t))$ be a sol^n to the RF w/ M^n closed. Then $\forall\ \alpha > 0$ and every $m \in \mathbb{N}$ $\exists$ a constant $C_m = C_m(m, n, \max\{\alpha, 1\})$ s.t. if

$$|Rm(x,t)|_{g(x,t)} \leq K \quad \forall\ x \in M^n,\ t \in \left[0, \frac{\alpha}{K}\right], \text{ then}$$

$$|\nabla^m Rm(x,t)|_{g(x,t)} \leq \frac{C_m K}{t^{m/2}} \quad \forall\ x \in M,\ t \in \left(0, \frac{\alpha}{K}\right].$$

<u>Remark</u> :- 1) the estimates follow parabolic rescaling.

2) The estimates deteriorates as $t \searrow 0$ which is the best which we can do w/o any other assumptions on g_0.

Corr :- (Long-time existence) If g_0 is a smooth metric on closed M^n, the unique sol^n $g(t)$ of the RF w/ $g(0)=g_0$ exists on a maximal time interval $0 \le t < T \le \infty$. Moreover, $T<\infty$ only if

$$\lim_{t \uparrow T} \left(\sup_{x \in M^n} |Rm(x,t)| \right) = \infty .$$

Let's start by computing the evolution of $|Rm|^2$ along the RF.

Lemma Along the RF

$$\partial_t |Rm|^2 = \Delta |Rm|^2 - 2|\nabla Rm|^2 + 2R^{ijkl}\left[R_{ijp}{}^{r}R_{r}{}^{p}{}_{kl} - 2R_{pik}{}^{r}R_{j}{}^{p}{}_{rl} + 2R_{pirl}R_{j}{}^{p}{}_{k}{}^{r}\right].$$

$$\therefore \partial_t |Rm|^2 \le \Delta |Rm|^2 - 2|\nabla Rm|^2 + C|Rm|^3$$

w/ $C = C(n)$.

Proof :- Recall that

$$\partial_t R_{ijk}{}^{l} = \Delta R_{ijk}{}^{l} + \left(R_{ijp}{}^{r}R_{r}{}^{p}{}_{k}{}^{l} - 2R_{pik}{}^{r}R_{j}{}^{p}{}_{r}{}^{l} + 2R_{pir}{}^{l}R_{j}{}^{p}{}_{k}{}^{r}\right)$$
$$- R_{ip}R^{p}{}_{jk}{}^{l} - R_{jp}R_{i}{}^{p}{}_{k}{}^{l} - R_{kp}R_{ij}{}^{pl} + R_{p}{}^{l}R_{ijk}{}^{p}$$

$$\therefore \partial_t (R_{ijkl}) = \partial_t (R_{ijk}{}^{m} g_{lm}) = \partial_t (R_{ijk}{}^{m}) g_{lm} + R_{ijk}{}^{m}(\partial_t g_{lm})$$

$$= \Delta R_{ijkl} + \Big(R_{ijp}{}^{r} R_{r}{}^{p}{}_{kl} - 2R_{pik}{}^{r} R_{j}{}^{p}{}_{rl} + 2R_{pirl} R_{j}{}^{p}{}_{k}{}^{r}$$

$$- R_{ip}R^{p}{}_{jkl} - R_{jp}R_{i}{}^{p}{}_{kl} - R_{kp}R_{ij}{}^{p}{}_{l} + R_{pl}R_{ijk}{}^{p} = - R_{pl}R_{ijk}{}^{p}$$

$$- 2R_{ijk}{}^{m} R_{lm}$$

$$\therefore\ \partial_t \left(|Rm|^2\right) = \partial_t \left(R_{ijkl} R_{abcd}\, g^{ia} g^{jb} g^{kc} g^{ld} \right)$$

$$= 2R^{ijkl}\partial_t(R_{ijkl}) + R_{ijkl}R_{a}{}^{jkl}\left(\partial_t g^{ia}\right) + R_{ijkl}R^{i}{}_{b}{}^{kl}\left(\partial_t g^{jb}\right)$$

$$+ R_{ijkl}R^{ij}{}_{c}{}^{l}\left(\partial_t g^{ck}\right) + R_{ijkl}R^{ijk}{}_{d}\left(\partial_t g^{ld}\right)$$

$$= 2R^{ijkl}\Delta R_{ijkl} + 2R^{ijkl}\Big(R_{ijp}{}^{r} R_{r}{}^{p}{}_{kl} - 2R_{pik}{}^{r} R_{j}{}^{p}{}_{rl} + 2R_{pirl} R_{j}{}^{p}{}_{k}{}^{r}\Big)$$

$$\cancel{- 2R^{ijkl}\left(R_{ip}R^{p}{}_{jkl} + R_{jp}R_{i}{}^{p}{}_{kl} + R_{kp}R_{ij}{}^{p}{}_{l} + R_{lp}R_{ijk}{}^{p} \right)}$$

$$\cancel{+ 2R_{ijkl}R^{ia}R_{ajkl} + \cdots}$$

Also, note $\Delta|Rm|^2 = \nabla^a \nabla_a \left(R_{ijkl} R^{ijkl} \right)$

$$= \nabla^a \left(2\left(\nabla_a R_{ijkl}\right) R^{ijkl} \right)$$

$$= 2R^{ijkl}\left(\Delta R_{ijkl}\right) + 2|\nabla Rm|^2$$

$\therefore$ we get $\partial_t |Rm|^2 = \Delta |Rm|^2 - 2|\nabla Rm|^2 + 2R^{ijkl}\Big[R_{ijp}{}^{r} R_{r}{}^{p}{}_{kl} - 2R_{pik}{}^{r} R_{j}{}^{p}{}_{rl}$

$$+ 2R_{pirl} R_{j}{}^{p}{}_{k}{}^{r} \Big].$$

<u>Cor.</u> (Doubling time estimate) $\exists\ c>0,\ c=c(n)$ s.t. for a sol^n $g(t)$ of the RF on $[0,T)$, we have

$$\sup_{x\in M} |Rm(x,t)|_{g(t)} \leq 2 \sup_{x\in M} |Rm(x,0)|_{g(x,0)}$$

$$\forall\ t\in \Big[0, \min\Big\{T, \frac{c}{\sup|Rm(x,0)|}\Big\}\Big\}.$$

<u>Remark</u> :- This shows why we can assume a bound for $|Rm|$ for a short time.

<u>Proof.</u> We have

$$\partial_t |Rm|^2 \leq \Delta |Rm|^2 - 2|\nabla Rm|^2 + C|Rm|^3 \quad \text{—①}$$

$$\leq \Delta |Rm|^2 + C|Rm|^3$$

now. $x \mapsto x^{3/2}$ is a locally Lipschitz function $\Rightarrow$ we can use the max. principle. Let $\sup_{(x,t)} |Rm(x,t)| = S$ then the PDE is

$$\partial_t S^2 \leq \Delta S^2 + CS^3$$

$\Rightarrow$ we need to solve the ODE $\dfrac{d\varphi}{dt} = \dfrac{c\varphi^2}{2}$

$$\Rightarrow \quad \frac{d\varphi}{\varphi^2} = \frac{c}{2}dt \quad \Rightarrow \Big[-\frac{1}{\varphi}\Big] = \frac{c}{2}t$$

$$\Rightarrow \frac{1}{\varphi(0)} - \frac{1}{\varphi(t)} = \frac{c}{2}t \quad \text{and so} \quad \varphi(t) = \frac{1}{\frac{1}{\varphi(0)} - \frac{c}{2}t}$$

$\therefore$ the max. principle tells us that

$$S(t) \le \frac{1}{\frac{1}{S(0)} - \frac{c}{2}t} \quad \text{as long as } t \in [0, T] \text{ satisfies } t < \frac{2}{cS(0)}.$$

For $c = \frac{1}{C}$ we get $S(t) \le \dfrac{2S(0)}{2 - CS(0)t} \le 2S(0)$

for $t \in [0, \min\{T, \frac{c}{S(0)}\}]$ and $c = c(n)$.

$\blacksquare$

<u>Proof of the global derivative estimates</u>

<u>Digression</u> :- Suppose we have a degree 1 tensor Q along the RF. Recall the formula for its derivative

$$\nabla_i Q_j = \frac{\partial}{\partial x^i} Q_j - \Gamma_{ij}^k Q_k$$

$$\Rightarrow \frac{\partial}{\partial t} \nabla_i Q_j = \frac{\partial}{\partial t}\left\{ \frac{\partial}{\partial x^i} Q_j - \Gamma_{ij}^k Q_k \right\}$$

$$= \frac{\partial}{\partial x^i}\frac{\partial}{\partial t} Q_j - \Gamma_{ij}^k \frac{\partial}{\partial t} Q_j - Q_k \frac{\partial}{\partial t}\Gamma_{ij}^k$$

$$= \nabla_i\left(\frac{\partial}{\partial t} Q\right)_j + \left(\nabla_i R_j{}^k + \nabla_j R_i{}^k - \nabla^k R_{ij}\right) Q_k.$$

Moreover,

$$\frac{\partial}{\partial t}|\nabla Q|^2 = \frac{\partial}{\partial t}\left(\nabla_i Q_j \nabla_k Q_l \, g^{ik} g^{jl}\right)$$

$$= 2\nabla^i Q^j \nabla_i \left(\frac{\partial}{\partial t} Q\right)_j + 2\nabla^i Q^j \left(\nabla_i R_j{}^k + \nabla_j R_i{}^k - \nabla^k R_{ij}\right) Q_k$$

$$+ 2R^{ik}\nabla_i Q^j \nabla_k Q_j + 2R^{jl}\nabla^i Q_j \nabla_i Q_l .$$

$\therefore$ when we take the time-derivative of quantities like $|\nabla Q|^2$, we should keep in mind the evolution of Q, Γ and g^{-1}.

<u>notation</u> :- $A * B$ will denote any quantity obtained from $A \otimes B$ by either summation over pairs of matching upper and lower indices or contraction by using g^{-1} or having constants depending only on n or $\mathrm{rank}(A)$, $\mathrm{rank}(B)$.

$\therefore$ e.g. in this convention

$$\partial_t |Rm|^2 = \Delta |Rm|^2 - 2|\nabla Rm|^2 + Rm * Rm * Rm .$$

We come back to the proof. The proof is by induction on m.

<u>$m=1$</u>:

$$\partial_t |\nabla Rm|^2 = 2\left\langle \nabla(\partial_t Rm), \nabla Rm\right\rangle + \underbrace{\nabla Ric * Rm * \nabla Rm}_{\text{from } \partial_t \Gamma}$$

$$+ Rc * \nabla Rm * \nabla Rm$$

from $(\partial_t g^{-1})$

$$\nabla(\partial_t Rm) = \nabla(\Delta Rm + Rm * Rm)$$
$$= \nabla(\Delta Rm) + Rm * \nabla Rm$$

also notice that for any tensor A, say

$$\nabla_i \nabla^a \nabla_a A = \nabla^a \nabla_i \nabla_a A + Rm * \nabla A$$
$$= \nabla^a(\nabla_a \nabla_i A + Rm * A) + Rm * \nabla A$$
$$= \Delta(\nabla A) + \nabla Rm * A + Rm * \nabla A$$

$$\therefore \quad \nabla(\partial_t Rm) = \Delta(\nabla Rm) + \nabla Rm * Rm + Rm * \nabla Rm$$

$\Rightarrow$

$$\partial_t |\nabla Rm|^2 = 2\left\langle \Delta(\nabla Rm) + \nabla Rm * Rm + Rm * \nabla Rm, \nabla Rm \right\rangle$$
$$+ \nabla Rm * Rm * \nabla Rm$$

$$= 2\langle \Delta(\nabla Rm), \nabla Rm\rangle + \nabla Rm * \nabla Rm * Rm$$

$$\Delta |A|^2 = \nabla^a \nabla_a \langle A, A\rangle = 2\nabla^a \langle \nabla_a A, A\rangle = 2A\Delta A + 2|\nabla A|^2$$

$$= \Delta|\nabla Rm|^2 - 2|\nabla^2 Rm|^2 + \nabla Rm * \nabla Rm * Rm$$

$\therefore$ we have the equation

$$\partial_t |\nabla Rm|^2 = \Delta |\nabla Rm|^2 - 2|\nabla^2 Rm|^2 + Rm * \nabla Rm * \nabla Rm$$
$$\leq \Delta |\nabla Rm|^2 - 2|\nabla^2 Rm|^2 + C|\nabla Rm|^2 |Rm|.$$

The problem w/ this is that we do not have any control on $|\nabla Rm|$ at $t=0$ for applying the maximum principle and we cannot get rid of the $|\nabla Rm|^2 |Rm|$ term b/c it is not negative.

we notice that in ① that $\partial_t |Rm|^2$ has $-2|\nabla Rm|^2$ term in it so if we can combine $|Rm|^2$ and $|\nabla Rm|^2$ term in such a way that the good terms from $\partial_t |Rm|^2$ take care of bad terms of $\partial_t |\nabla Rm|^2$ then we'll be in business.

Define the function

$$F_1 = t|\nabla Rm|^2 + \beta |Rm|^2$$ w/ β a constant to be chosen later.

note if $g \mapsto \lambda^2 g$ then $|Rm|^2 \mapsto \lambda^{-4}|Rm|^2$

and $|\nabla Rm|^2 \mapsto \lambda^{-6}|\nabla Rm|^2$ so we multiply $|\nabla Rm|^2$ by a factor of $t \mapsto (\text{dist})^2$ so as to make the function F_1 "dimensionless".

note :- $F_1(x,0) = \beta |Rm|^2(0) \leq \beta K^2$.

then
$$\begin{aligned} \partial_t F_1 &= t\,\partial_t |\nabla Rm|^2 + |\nabla Rm|^2 + \beta\,\partial_t |Rm|^2 \\ &\leq t\left(\Delta|\nabla Rm|^2 - 2|\nabla^2 Rm|^2 + C|\nabla Rm|^2 |Rm|^2\right) + |\nabla Rm|^2 \\ &\quad + \beta\left(\Delta |Rm|^2 - 2|\nabla Rm|^2 + C|Rm|^3\right) \end{aligned}$$

$$= \Delta\left(t|\nabla Rm|^2 + \beta|Rm|^2\right) + \left(1 + Ct|Rm| - 2\beta\right)|\nabla Rm|^2 + C\beta|Rm|^3$$ (w/ all constants depending on n only)

$\because$ $|Rm| \le K$ $\forall t \in [0, \frac{\alpha}{K}] \Rightarrow Ct|Rm| \le C\alpha$

$$\le \Delta F + (1 + C\alpha - 2\beta)|\nabla Rm|^2 + C\beta K^3$$

Choose $\beta \ge \frac{(1+C\alpha)}{2}$, $\beta = \beta(n, \max\{\alpha, 1\})$ to get

$$\partial_t F \le \Delta F + C\beta K^3.$$

By max. principle, we have to find the sol^n of the ODE

$$\frac{d\varphi}{dt} = C\beta K^3, \quad \varphi(0) = \beta K^2$$

$$\Rightarrow \quad \varphi(t) = C\beta K^3 t + \beta K^2$$

$$\Rightarrow \quad t|\nabla Rm|^2 \le \beta K^2 + C\beta K^3 t \le (1 + C\alpha)\beta K^2 \le CK^2$$

w/ $C = C(n, \max\{\alpha, 1\})$.

$\therefore$ $|\nabla Rm|^2 \le \frac{CK^2}{t} \Rightarrow |\nabla Rm| \le \frac{CK}{t}$ which proves the base step for induction.

Assume that we have the desired estimate for $|\nabla^j Rm|$ $\forall\ 1 \le j < m$.

We want to prove the estimate for $j = m$.

$$\partial_t |\nabla^m Rm|^2 = 2\left\langle \partial_t (\nabla^m Rm), \nabla^m Rm \right\rangle + Rc * (\nabla^m Rm * \nabla^m Rm)$$

$$\partial_t (\nabla^m Rm) = \nabla^m (\partial_t Rm) + \sum_{j=0}^{m-1} \nabla^j (\nabla Rc * \nabla^{m-j} Rm)$$

$$= \nabla^m (\Delta Rm + Rm * Rm) + \sum_{j=0}^{m} \nabla^j Rm * \nabla^{k-j} Rm$$

$$= \nabla^m \Delta Rm + \sum_{j=0}^{m} \nabla^j Rm * \nabla^{k-j} Rm$$

$$\nabla^m \Delta Rm = \Delta \nabla^m Rm + \sum_{j=0}^{m} \nabla^j Rm * \nabla^{m-j} Rm$$

e.g. $$\nabla^2 \nabla_i \nabla^i Rm = \nabla(\nabla_i \nabla \nabla^i Rm + Rc * \nabla Rm)$$

$$= \nabla_i \nabla \nabla \nabla^i Rm + Rc * \nabla^2 Rm + \nabla Rc * \nabla Rm$$

$$= \nabla_i \nabla (\nabla^i \nabla Rm + Rc * Rm) + Rc * \nabla^2 Rm + \nabla Rc * \nabla Rm$$

$$= \nabla_i \nabla \nabla^i \nabla Rm + \nabla^2 Rc * Rm + Rc * \nabla^2 Rm + \nabla Rc * \nabla Rm$$

$$= \Delta \nabla^2 Rm + \nabla_i (Rc * \nabla Rm) + \nabla^2 Rc * Rm + Rc * \nabla^2 Rm + \nabla Rc * \nabla Rm$$

$$= \Delta \nabla^2 Rm + Rc * \nabla^2 Rm + \nabla Rc * \nabla Rm + \nabla^2 Rc * Rm$$

and so on.

$$= \Delta \nabla^m Rm + \sum_{j=0}^{m} \nabla^j Rm * \nabla^{m-j} Rm$$

$$\therefore \quad \partial_t |\nabla^m Rm|^2 = 2\left\langle \Delta \nabla^m Rm + \sum_{j=0}^{m} \nabla^j Rm * \nabla^{m-j} Rm, \nabla^m Rm \right\rangle$$

$$+ Rc * \nabla^m Rm * \nabla^m Rm$$

and since $2A\Delta A = \Delta|A|^2 - 2|\nabla A|^2$ for any tensor A, using $A = \nabla^m Rm$

$$= \Delta|\nabla^m Rm|^2 - 2|\nabla^{m+1} Rm|^2 + \sum_{j=0}^{m} \nabla^j Rm * \nabla^{m-j} Rm * \nabla^m Rm \quad \text{--- } (2).$$

$$\Rightarrow \partial_t |\nabla^m Rm|^2 \leq \Delta|\nabla^m Rm|^2 - 2|\nabla^{m+1} Rm|^2 + \sum_{j=0}^{m} c_{m,j} \, |\nabla^j Rm| \cdot |\nabla^{m-j} Rm| \cdot |\nabla^m Rm|$$

w/ $c_{m,j} = c_{m,j}(n, m, j)$. $\quad$ --- (3)

Using induction hypothesis we get

$$\partial_t |\nabla^m Rm|^2 \leq \Delta|\nabla^m Rm|^2 - 2|\nabla^{m+1} Rm|^2 + (c_{m,0} + c_{m,m}) K |\nabla^m Rm|^2$$

$$+ \sum_{j=1}^{m-1} c_{m,j} \frac{C_j}{t^{j/2}} \frac{C_{m-j}}{t^{\frac{(m-j)}{2}}} K^2 |\nabla^m Rm|$$

$$\leq \Delta|\nabla^m Rm|^2 - 2|\nabla^{m+1} Rm|^2 + K\left(C_m' |\nabla^m Rm|^2 + \frac{C_m''}{t^{m/2}} K |\nabla^m Rm|\right)$$

w/ $t \in \left(0, \frac{\alpha}{K}\right]$, C_m' and C_m'' depend only on m and n

Using the Young's inequality on $\frac{C_m''}{t^{m/2}} K |\nabla^m Rm| \leq \frac{C_m''^2 |\nabla^m Rm|^2}{2} + \frac{K^2}{2t^m}$

$$\therefore \partial_t |\nabla^m Rm|^2 \leq \Delta |\nabla^m Rm|^2 - 2|\nabla^{m+1} Rm|^2 + \tilde{C}_m K \left(|\nabla^m Rm|^2 + \frac{K^2}{t^m} \right) \qquad \text{--- (4).}$$

Define the function

$$F_m = t^m |\nabla^m Rm|^2 + \beta_m \sum_{k=1}^{m} \frac{(m-1)!}{(m-k)!} t^{m-k} |\nabla^{m-k} Rm|^2$$

w/ β_m to be chosen later.

note:- $F_m(0) = \beta_m \frac{(m-1)!}{0!} |Rm|^2 \leq \beta_m (m-1)! K^2.$

Also, by the inductive hypothesis, $\exists$ constants $\tilde{C}_k = \tilde{C}_k(k,n)$ s.t.

$$\forall \ 1 \leq k < m, \ \partial_t |\nabla^k Rm|^2 \leq \Delta |\nabla^k Rm|^2 - 2|\nabla^{k+1} Rm|^2 + \frac{\tilde{C}_k K^3}{t^k}$$

$$\text{--- (5).}$$

$$\therefore \partial_t F_m = t^m \partial_t |\nabla^m Rm|^2 + m t^{m-1} |\nabla^m Rm|^2 + \beta_m \sum_{k=1}^{m} \frac{(m-1)!}{(m-k)!} (m-k) t^{m-k-1} |\nabla^{m-k} Rm|^2$$
$$+ \beta_m \sum_{k=1}^{m} \frac{(m-1)!}{(m-k)!} t^{m-k} \partial_t |\nabla^{m-k} Rm|^2$$

$$\leq t^m \Delta |\nabla^m Rm|^2 - 2t^m |\nabla^{m+1} Rm|^2 + \tilde{C}_m t^m K |\nabla^m Rm|^2$$

$$+ \tilde{C}_m K^3 + m t^{m-1} |\nabla^m Rm|^2 + \beta_m \sum_{k=1}^{m} \frac{(m-1)!}{(m-k)!} (m-k) t^{m-k-1} |\nabla^{m-k} Rm|^2$$

$$+ \beta_m \sum_{k=1}^{m} \frac{(m-1)!}{(m-k)!} t^{m-k} \left(\Delta |\nabla^{m-k} Rm|^2 - 2 |\nabla^{m-k+1} Rm|^2 + \frac{\tilde{C}_{m-k} K^3}{t^{m-k}} \right)$$

$$\leq \Delta F_m + |\nabla^m Rm|^2 \left(\tilde{C}_m t^{m-1} \alpha + m t^{m-1} - 2\beta_m \frac{(m-1)!}{(m-1)!} t^{m-1} \right)$$

$$+ \tilde{C}_m K^3 + \beta_m \sum_{k=1}^{m} \frac{(m-1)!}{(m-k)!} \left\{ (m-k) t^{m-k-1} |\nabla^{m-k} Rm|^2 + \tilde{C}_{m-k} K^3 \right\}$$

$$- 2 \beta_m \sum_{k=2}^{m} \frac{(m-1)!}{(m-k)!} t^{m-k} |\nabla^{m-k+1} Rm|^2$$

$$\leq \Delta F_m + |\nabla^m Rm|^2 t^{m-1} \left(\tilde{C}_m \alpha + m - 2\beta_m \right)$$

$$+ \left(\tilde{C}_m + \beta_m \tilde{C}'_m \right) K^3 + \beta_m \sum_{k=1}^{m} \frac{(m-1)!}{(m-k-1)!} t^{m-k-1} |\nabla^{m-k} Rm|^2$$

$$- 2 \beta_m \sum_{k=2}^{m} \frac{(m-1)!}{(m-k)!} t^{m-k} |\nabla^{m-k+1} Rm|^2$$

w/ $\tilde{C}'_m = \sum_{k=1}^{m} \frac{(m-1)!}{(m-k)!} \tilde{C}_{m-k}$.

= this term is negative. In fact, the def^n of

F_m was chosen so that the good terms $-\dfrac{2(m-1)!}{(m-k)!} t^{m-k} |\nabla^{m-k+1} Rm|^2$

Obtained by differentiating $|\nabla^{m-k} Rm|^2$ compensate for the bad terms

$\dfrac{(m-1)!}{(m-k+1)!} (m-k+1) t^{m-k} |\nabla^{m-k+1} Rm|^2$ which are obtained when

we differentiate $t^{m-(k-1)}$ term.

$\therefore$ we get

$$\partial_t F_m \leq \Delta F_m + (\tilde{C}_m \alpha + m - 2\beta_m) t^{m-1} |\nabla^m Rm|^2 + (\tilde{C}_m + \tilde{C}_m' \beta_m) K^3$$

Choosing $\beta_m \geq \dfrac{\tilde{C}_m \alpha + m}{2}$ gives that $\forall\, t \in [0, \frac{\alpha}{K}]$

$$\partial_t F_m \leq \Delta F_m + (\tilde{C}_m + \tilde{C}_m' \beta_m) K^3$$

$\Rightarrow$ By the max. principle

$$F_m(x,t) \leq (\tilde{C}_m + \tilde{C}_m \beta_m) K^3 t + \beta_m (m-1)!\, K^2$$

$$\leq C_m^2 K^2$$

$$\Rightarrow \quad |\nabla^m Rm| \leq \frac{C_m K}{t^{m/2}} \qquad \forall\; 0 < t \leq \frac{\alpha}{K}$$

$\boxed{}$